COVARIANT DERIVATIVES AND CONNECTIONS

MORE NEXT WEEK

COVARIANT DERIVATIVE ∇_μ $(;)$

COVARIANT

BASIS INDEPENDENT

$\nabla_\mu \phi(x) \Rightarrow$ vector ✓

$\nabla_\mu V^\nu$ IS A TENSOR ✓

and so on

LEVI-CIVITA CONNECTION $\Gamma_{\alpha\beta}^{\ \ \gamma}$ $(g_{\mu\nu})$

$\nabla_\mu = ?$

It looks like ∂_μ

in flat space

(and inertial coord's) cartesian c.

and/or

it transforms as

a tensor (otherwise) curved space

\+ Properties

→ Linearity

→ Leibniz rule ("chain rule")

$(\ \ )_{\alpha\beta}^{\ \ \gamma} \equiv \Gamma_{\alpha\beta}^{\ \ \gamma}$

CHRISTOFFEL SYMBOLS

NOT TENSORS

DON'T FOLLOW THE TENSOR TRANSFORMATION RULES

\+ Properties (torsion)

$\Gamma_{\alpha\beta}^{\ \ \gamma}$ MAY / MAY NOT depend on $g_{\mu\nu}$

BUT

Given $g_{\mu\nu}$ $\xrightarrow{\nabla_\beta g_{\mu\nu}=0}$ $\Gamma_{\alpha\beta}^{\ \ \gamma}$ unique

metricity condition.

\+ example.

MOTIVATION

MATHS

2. DERIVATIVES

∂_μ $(,)$

$\partial_\mu \phi(x) \equiv \phi(x)_{,\mu}$ VECTOR ✓

$\partial_\mu V^\nu \equiv V^\nu_{\ ,\mu}$ NOT A TENSOR

DOES NOT TRANSFORM AS TENSOR DOES

BASIS DEPENDENT

Physical quantities do not depend on basis

velocity of an object should be the same if you use cartesians or polar coordinates!

PHYSICS

1. CONSERVED QUANTITIES or how they evolve

$\partial_\mu T^{\mu\nu} = 0$ FLAT SPACE $\xrightarrow{\text{GENERALIZE}}$ $(\ \ )_\mu T^{\mu\nu} = 0$ CURVED SPACE

① COVARIANT DERIVATIVE ∇_μ

1.1 HOW DOES ∇_μ LOOK LIKE?

$\phi \xrightarrow{\nabla_\mu}$ ~~vector~~ $\nabla_\mu \phi$

$(0,0) \longrightarrow (0,1)$ (dual-v) 1-form

$V^\nu \xrightarrow{\nabla_\mu}$ tensor $\nabla_\mu V^\nu$

$(1,0) \longrightarrow (1,1)$

∇_μ map:

upper ↑ lower ↓

$(k,\ell) \longrightarrow (k, \ell+1)$

TENSOR FIELDS → TENSOR FIELDS

Requirements: (In order to be a derivative)

(1) Linearity $\quad \nabla(\alpha T + \beta S) = \alpha \nabla T + \beta \nabla S \quad , \quad \alpha, \beta \equiv \text{const}$

(2) Leibniz rule $\quad \nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$

($\otimes$: tensor product)

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu \qquad \omega_{\mu'} = \frac{\partial x^\nu}{\partial \tilde{x}^{\mu'}} \omega_\nu$$

Expression:

$$\boxed{\nabla_\mu = \partial_\mu + \text{something which guarantees covariance}}$$

• Vector:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + (\ \ )^{\nu}_{\mu\lambda} V^\lambda \quad (*) \quad \equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

↑ who?

Covariance:

$\{x^\mu\}$ basis $\longrightarrow \{\tilde{x}^{\mu'}\}$ basis $\Rightarrow$ coord. transf.

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial \tilde{x}^{\mu'}} \frac{\partial \tilde{x}^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu \quad [A]$$

(Recall how x_μ and x^μ transforms)

$$\nabla_{\mu'} V^{\nu'} \overset{(*)}{=} \underbrace{\partial_{\mu'} V^{\nu'}}_{} + (\Gamma)^{\nu'}_{\mu'\lambda'} V^{\lambda'} \quad \overset{\text{change of coord's}}{=}$$

we should know how they transform (SYMBOLS)

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \underbrace{\frac{\partial}{\partial x^\mu}}_{\partial_\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda \quad [B]$$

Covariance argument $\Rightarrow [A] \equiv [B]$

$$\hookrightarrow \Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial \tilde{x}^{\mu'}} \frac{\partial x^\lambda}{\partial \tilde{x}^{\lambda'}} \frac{\partial \tilde{x}^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial \tilde{x}^{\mu'}} \frac{\partial x^\lambda}{\partial \tilde{x}^{\lambda'}} \frac{\partial^2 \tilde{x}^{\nu'}}{\partial x^\mu \partial x^\lambda}$$

SYMBOL

free indices — NO tensor

1.2 Then :

* SCALARS $\nabla_\mu \phi(x) = \partial_\mu \phi(x)$

* VECTORS $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\lambda} V^\lambda$

Not only raising or lowering indices since $\Gamma^{\rho}_{\mu\nu}$ is a symbol.

* 1-FORMS $\nabla_\mu \omega_\nu = \partial_\mu W_\nu - \Gamma^{\lambda}_{\mu\nu}\omega_\lambda$

* General tensor (look it up (eq 3.17) Page 97 Carroll, Wald)

$\boxed{\nabla_\rho g_{\mu\nu} = 0}$ Metricity condition expression

1.3. Property : ∇_μ commutes with constractions :

$T^{\mu}_{\ \nu\rho} \to \nabla_\mu (T^{\lambda}_{\ \lambda\rho}) \equiv$ 1st $\nabla_\mu T^{\lambda}_{\ \nu\rho}$ + 2nd contraction λ, ν

Divergence : $\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|}\, V^\mu)$

ASSOCIATED TO A METRIC

② CONNECTION $\Gamma^{\gamma}_{\alpha\beta}$, ∇_μ

$\boxed{g_{\mu\nu} \Rightarrow \Gamma^{\gamma}_{\alpha\beta} \text{ unique} \Rightarrow \nabla_\mu \text{ unique Levi-Civita C.}}$

Name connection comes from the fact it is used to "transport" vectors from the tangent space to the dual.

(1) Metricity condition
+ symmetric permutations
& $g^{\sigma\rho}$

$- \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0$
$+ \nabla_\mu g_{\nu\rho} = \dots$
$+ \nabla_\nu g_{\rho\mu} = \dots$

$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\tau} [\partial_\mu g_{\tau\nu} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu}]}$$

(2) Property : Torsion free $\Rightarrow$ $\boxed{\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}}$ regardless symmetry

$T^{\lambda}_{\ \mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 2\Gamma^{\lambda}_{[\mu\nu]}$ antisymmetric in lower indices
∇_μ symmetric in lower index
$\Rightarrow$ Torsion free.

At the end.

$\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f$

EXAMPLE

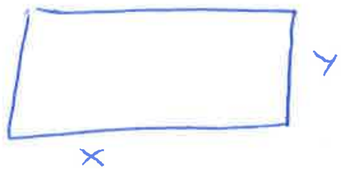
$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right]$$

2D Flat space

⊛ CARTESIAN COORDINATES

$ds^2 = 1\,dx^2 + 1\,dy^2$

$g_{ij} = \begin{bmatrix}1 & \\ & 1\end{bmatrix}$; $g^{ij} = \begin{bmatrix}1 & \\ & 1\end{bmatrix}$



NO NEED FOR CALCULATIONS

$\boxed{\Gamma^{k}_{ij} = 0}$ ⇐ ∇_μ is constructed such that ∂_μ is recovered in flat space and inertial coord's.

CHANGE OF COORD'S

⊛ POLAR COORDINATES

$ds^2 = dr^2 + r^2 d\theta^2$

$g_{ij} = \begin{bmatrix}1 & \\ & r^2\end{bmatrix}$; $g^{ij} = \begin{bmatrix}1 & \\ & 1/r^2\end{bmatrix}$

$i, j, k = \{r, \theta\}$

$k = r$
↳ $i = r$
  ↳ $j = r$
  ↳ $j = \theta$
↳ $i = \theta$
  ↳ $j = r$
  ↳ $j = \theta$
$k = \theta$ (same)

$\Gamma^{r}_{rr} = 0$

$\Gamma^{r}_{r\theta} \stackrel{\text{sym}}{=} \Gamma^{r}_{\theta r} = 0$

$\Gamma^{r}_{\theta\theta} = \frac{1}{2} g^{r\rho}\left(\partial_\theta g_{\theta\rho} + \partial_\theta g_{\rho\theta} - \partial_\rho g_{\theta\theta}\right) =$ ρ sum over r & θ

$= \frac{1}{2} g^{rr}\left(\underbrace{\partial_\theta g_{\theta r}}_{=0} + \underbrace{\partial_\theta g_{r\theta}}_{=0} - \partial_r g_{\theta\theta}\right) +$

$+ \frac{1}{2} \underbrace{g^{r\theta}}_{=0}\left(\partial_\theta g_{\theta\theta} + \partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta}\right)$

$= \frac{1}{2}\cdot 1\left(-\partial_r r^2\right) = -r$

$\Gamma^{\theta}_{rr} = 0$

$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = 1/r$

$\Gamma^{\theta}_{\theta\theta} = 0$

Torsion - free demonstration?