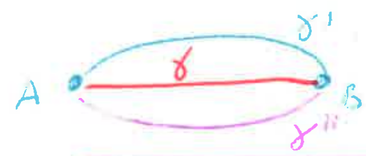
GEODESICS

* We need to describe trajectories
* We need to measure distances.

HOW TO ~~MEASU~~ MEASURE DISTANCES?

TRAJECTORIES $\Rightarrow$ GEODESIC



Parallel transport

Trajectory which minimizes the path

Length + variational principle.

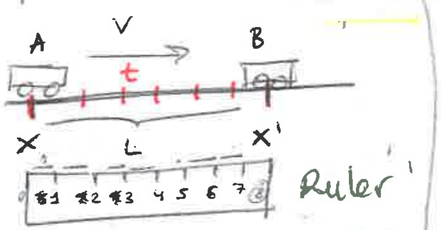
$\Downarrow$

$\delta L = 0$ (Wald)

$\to$ (Geod. eq.)

$$\boxed{\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = 0}$$

FLAT S-T



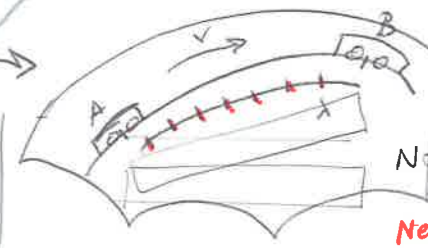
Ruler

Parametrize trajectory by using the parameter time.

$dx = \frac{dx}{dt}dt = \dot{x}dt$

$$\int dx = \int v\, dt = L$$

CURVED S-T



Not real distance

New ruler

What do we have?

Metric! $g_{\mu\nu}$

"Line element"

$ds^2 = g_{\mu\nu}dx^\mu dx^\nu =$

$= g_{00}dt^2 + g_{ij}dx^i dx^j$

$$ds = \sqrt{|g_{\mu\nu}dx^\mu dx^\nu|}$$

λ = afine parameter of the curve.

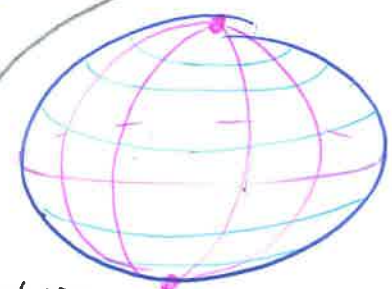
$$dx^\mu = \frac{dx^\mu}{d\lambda}d\lambda \equiv \dot{x}^\mu d\lambda$$

$\Downarrow$

$$\boxed{L = \int \sqrt{|g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu|}\, d\lambda}$$

"Length" in curbed space-time

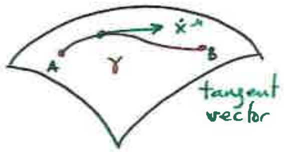
($\Leftrightarrow$ action)



Meridians and Equator are Geodesics

(Parallels in general are not) ex.

In order to measure distances on the surface:



Curve γ of length L.

τ : affin parameter of the curve

x^μ, $g^{\mu\nu}(x)$, $(dl)^2$

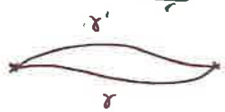
$$\left.\begin{aligned}(dl)^2 &= g_{\mu\nu}(x)\, dx^\mu dx^\nu \\ dx^\mu &= \frac{dx^\mu}{d\tau} d\tau = \dot{x}^\mu d\tau\end{aligned}\right\} \quad \boxed{L = \int \sqrt{g_{\mu\nu}(x(\tau))\, \dot{x}^\mu \dot{x}^\nu}\, d\tau}$$

LENGTH = ACTION $\rightsquigarrow$ $\boxed{S \underset{v \ll c}{\sim} \int \left(\tfrac{1}{2} m v^2 + \tfrac{GM}{r}\right) dt}$

L = shortest distance from A to B. $g_{tt} - v^2$ $\quad 1 + \frac{2GM}{r}$ $\quad 1 + \frac{2GM}{r} - v^2$ $\quad \frac{1}{\gamma} \approx 1$

$\Downarrow$

$\delta L = L(\gamma) - L(\gamma') = 0$



There is only one shortest path.

$$0 = \delta L \underset{\boxed{IRS}}{=} \int \delta\left(\sqrt{g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu}\right) d\tau = \int \frac{1}{2} \frac{1}{\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} \left(\overbrace{(\delta g_{\mu\nu})}^{(\partial_\sigma g_{\mu\nu})\delta x^\sigma} \dot{x}^\mu \dot{x}^\nu + 2 g_{\mu\nu} \overbrace{(\delta \dot{x}^\mu)}^{\delta \frac{dx^\mu}{d\tau} = \frac{d}{d\tau}\delta x^\mu} \dot{x}^\nu \right) d\tau =$$

$g_{\mu\nu} = g_{\nu\mu}$

$$= \frac{1}{2} \int \left[(\partial_\sigma g_{\mu\nu})\, \delta x^\sigma \dot{x}^\mu \dot{x}^\nu + 2 g_{\mu\nu} \left(\frac{d}{d\tau} \delta x^\mu\right) \dot{x}^\nu \right] d\tau =$$

For simplicity

$\tau \equiv$ fixed $\Rightarrow \dot{x}^\mu = (1, \vec{0})$

$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = |\dot{x}^\mu|^2 = 1$

$$\frac{d}{d\tau}(g_{\mu\nu}\, \delta x^\mu \dot{x}^\nu) = \frac{d}{d\tau}(g_{\mu\nu} \dot{x}^\nu)\, \delta x^\mu + g_{\mu\nu} \left(\frac{d}{d\tau} \delta x^\mu\right) \dot{x}^\nu$$

$$= \frac{1}{2} \int \left[(\partial_\sigma g_{\mu\nu})\, \dot{x}^\mu \dot{x}^\nu\, \delta x^\sigma - 2 \frac{d}{d\tau}(g_{\mu\nu} \dot{x}^\nu)\, \delta x^\mu \right] d\tau =$$

$$= \frac{1}{2} \int \underbrace{\left[(\partial_\sigma g_{\mu\nu})\, \dot{x}^\mu \dot{x}^\nu - 2 \frac{d}{d\tau}(g_{\sigma\nu} \dot{x}^\nu) \right]}_{=0} \delta x^\sigma d\tau \equiv 0$$

$$\frac{d}{d\tau}\Big(g_{\mu\nu} \underset{\dot{x}^\mu \delta\tau}{\underbrace{\delta x^\mu}} \dot{x}^\nu = \underbrace{g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu}_{1\ (IRS)} \delta\tau\Big) = \frac{d}{d\tau} 1\, \delta\tau = d1 = 0 \ ???$$

$$\Rightarrow \frac{1}{2} \partial_\sigma g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu - \Big(\overbrace{\frac{d}{d\tau} g_{\sigma\nu}}^{\partial_\mu g_{\sigma\nu}\, \dot{x}^\mu \dot{x}^\nu}\Big) \dot{x}^\nu - g_{\sigma\nu}\, \ddot{x}^\nu = 0$$

$$2 g_{\sigma\nu}\, \ddot{x}^\nu + \overbrace{2 \partial_\mu g_{\sigma\nu}\, \dot{x}^\mu \dot{x}^\nu}^{g_{\sigma\nu} = g_{\nu\sigma}\ \to (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma})\dot{x}^\mu \dot{x}^\nu} - \partial_\sigma g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = 0$$

$\frac{1}{2} g^{\sigma\nu}$

$$\ddot{x}^\nu + \frac{1}{2} g^{\sigma\nu} \left[\partial_\nu g_{\sigma\nu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} \right] \dot{x}^\mu \dot{x}^\nu = 0 \qquad \boxed{g_{\sigma\nu}^{-1} = g^{\sigma\nu}}$$

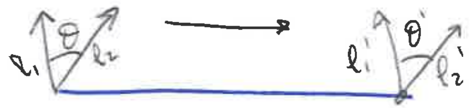
$\Downarrow$

$$\boxed{\ddot{x}^\sigma + \Gamma^\sigma_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = 0} \quad \text{Geodesic equation.}$$

$$\boxed{\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\nu} \left[\partial_\mu g_{\sigma\nu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} \right]} \quad \text{Levi-Civita connection.}$$

$$\boxed{\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}}$$

Parallel transport



LENGTHS AND
ANGLES ARE
PRESERVED

$\ell_1 = \ell_1'$
$\ell_2 = \ell_2'$
$\theta = \theta'$

Equation: Notation

$\nabla_{\dot\gamma} \dot\gamma|_A = 0$ → The variation of $\dot\gamma|_A$ (vector) along the curve γ is null.

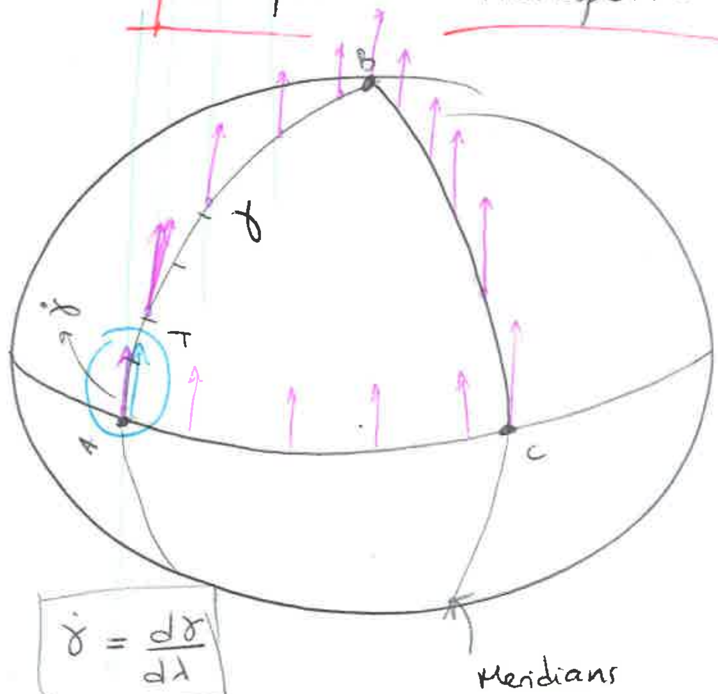
↳ defined by the flow $\dot\gamma$

Could be written: (in coord's)

$$\dot x^\alpha (\nabla_\alpha \dot x)^\beta = 0$$

$$\boxed{\dot x^\alpha \left[\partial_\alpha \dot x^\beta + \Gamma^\beta_{\alpha\delta}\, \dot x^\delta\right] = 0}$$

⇒ Geodesics should satisfy the parallel transport.



$$\boxed{\dot\gamma = \frac{d\gamma}{d\lambda}}$$

$\dot\gamma$ remains unchanged.

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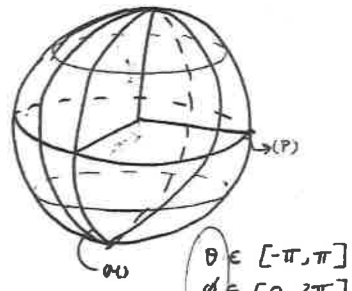
Demonstrate ~~whether~~ meridians and parallels are or not geodesics.



$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta \, d\phi^2$$

$$g_{\mu\nu} = a^2 \begin{bmatrix} 1 & \\ & \sin^2\theta \end{bmatrix}$$

$$g^{\mu\nu} = \frac{1}{a^2} \begin{bmatrix} 1 & \\ & \frac{1}{\sin^2\theta} \end{bmatrix}$$

$\theta \in [-\pi, \pi]$

$\phi \in [0, 2\pi]$

First of all, we calculate the Christoffel symbols:

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} [\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}]$$

$\Gamma^{\theta}_{\theta\theta} = \Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = 0, \quad \Gamma^{\phi}_{\theta\theta} = \Gamma^{\phi}_{\phi\phi} = 0$

$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \quad , \quad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{1}{\tan\theta}$

The geodesics are given by

$$\ddot{x}^\mu + \Gamma^{\mu}_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0 \qquad \dot{} = \frac{d}{d\tau}, \quad \tau = \text{affin parameter}$$

$\mu = \theta, \phi.$

$$\left.\begin{aligned} &\ddot{\theta} + \Gamma^{\theta}_{\phi\phi} \dot{\phi}^2 = 0 \\ &\ddot{\phi} + 2\Gamma^{\phi}_{\phi\theta} \dot{\phi}\dot{\theta} = 0 \end{aligned}\right\} \quad \begin{aligned} &\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \\ &\ddot{\phi} + 2\cot\theta\,\dot{\phi}\dot{\theta} = 0 \end{aligned}$$

Let's show if meridians and parallels fullfill these equations. Before of that, we considere the gauge:

$|\dot{x}^\mu| = 1 \Rightarrow \dot{\theta}^2 + \dot{\phi}^2 = 1$, along the curve (not instantaneously).

Meridians:

$\phi \equiv$ const.

$\theta = \theta(\tau)$

$\ddot{\phi} + 2\cot\theta\,\dot{\phi}\dot{\theta} = 0 + 0 = 0$

$\dot{\phi} = 0$

$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = \ddot{\theta} - 0 = 0 - 0 = 0$

but $\dot{\theta} = \sqrt{1 - \dot{\phi}^2} = 1$

$\boxed{\text{Meridians are geodesics!}}$

Parallels

$\boxed{\begin{aligned} &\theta \equiv \text{const.} \\ &\phi = \phi(\tau) \end{aligned}}$

$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 - \sin\theta\cos\theta \equiv 0$ if $\theta = \pi/2$

$\dot{\theta} = 0$

$\dot{\phi}^2 = 1 - \dot{\theta} = 1$

$\ddot{\phi} + 2\cot\theta\,\dot{\phi}\dot{\theta} = \ddot{\phi} + 0 = 0 + 0 = 0$

$\boxed{\begin{aligned} &\text{Parallels are not geodesics,} \\ &\text{Equator is a geodesic } (\theta = 0) \end{aligned}}$

$\dot{x}^\alpha \nabla_\alpha \dot{x}^\rho =$

$= \dot{x}^\alpha [\partial_\alpha \dot{x}^\rho + \Gamma^{\rho}_{\alpha\delta} \dot{x}^\delta] = 0$

$\rho = \theta$

$\dot{\phi} \nabla_\phi \dot{x}^\rho = \dot{\phi} (\partial_\phi \dot{\theta} + \Gamma^{\theta}_{\phi\alpha} \dot{x}^\alpha)$

$\dot{\phi} \nabla_\phi \dot{\phi} = \dot{\phi} [\partial_\phi \dot{\theta} + \Gamma^{\phi}_{\phi\alpha} \dot{x}^\alpha] = \dot{\phi} [\underbrace{\partial_\phi \dot{\theta}}_{0} + \Gamma^{\theta}_{\phi\phi} \dot{\phi}] = -\dot{\phi} \sin\theta \cos\theta \, \dot{\phi} = 0$ if $\theta = \pi/2$ equator

$\rho = \theta$

$|\dot{x}^\mu| = 1$

$$x^\mu(\tau) = \begin{pmatrix} \theta_0 \\ \tau \end{pmatrix} \longrightarrow x = \begin{pmatrix} s\tau \end{pmatrix}$$

$$\dot{x}^\mu = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \dot{x}'^\mu = \begin{pmatrix} 0 \\ \frac{1}{\sin^2\theta_0} \end{pmatrix}$$

$$|\dot{x}^\mu| = g_{\mu\nu} \dot{x}^\nu = g_{11} \dot{x}^1 + g_{22} \dot{x}^2 = 0 + \sin^2\theta \equiv 1$$

It does not work

It is a matter of normalization (affin parameter)

$$\dot{x}^2 = \frac{1}{\sin^2\theta_0} = s$$

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2-sphere
radius a

$$g_{\mu\nu}(\theta,\phi) = a^2\begin{bmatrix} 1 & \\ & \sin^2\theta \end{bmatrix} ; \quad g^{\mu\nu} = \frac{1}{a^2}\begin{bmatrix} 1 & \\ & \frac{1}{\sin^2\theta} \end{bmatrix}$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot an\,\theta$$

Ricci scalar:

* $$R_{\mu\nu\rho}{}^{\sigma} = -2\partial_{[\mu}\Gamma^{\sigma}_{\nu]\rho} + 2\Gamma^{\lambda}_{\rho[\mu}\Gamma^{\sigma}_{\nu]\lambda} =$$

$$= \partial_\nu\Gamma^{\sigma}_{\mu\rho} - \partial_\mu\Gamma^{\sigma}_{\nu\rho} + \Gamma^{\lambda}_{\rho\mu}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\lambda}_{\rho\nu}\Gamma^{\sigma}_{\mu\lambda}$$

* $$R_{\mu\alpha\rho}{}^{\alpha} = \partial_\alpha\Gamma^{\alpha}_{\mu\rho} - \partial_\mu\Gamma^{\alpha}_{\alpha\rho} + \Gamma^{\lambda}_{\rho\mu}\Gamma^{\alpha}_{\alpha\lambda} - \Gamma^{\lambda}_{\rho\alpha}\Gamma^{\alpha}_{\mu\lambda} = R_{\mu\rho}$$

* $$R^{\mu}{}_{\mu} = g^{\rho\mu}R_{\mu\rho} =$$

$$= g^{\rho\mu}\left[\partial_\alpha\Gamma^{\alpha}_{\mu\rho} - \partial_\mu\Gamma^{\alpha}_{\alpha\rho} + \Gamma^{\lambda}_{\rho\mu}\Gamma^{\alpha}_{\alpha\lambda} - \Gamma^{\lambda}_{\rho\alpha}\Gamma^{\alpha}_{\mu\lambda}\right] = R$$

$\mu,\nu,\rho\ldots = \theta,\phi$
$(\rho\equiv\mu)$

$$R = g^{\mu\mu}\left[\partial_\alpha\Gamma^{\alpha}_{\mu\mu} - \partial_\mu\Gamma^{\alpha}_{\alpha\mu} + \Gamma^{\lambda}_{\mu\mu}\Gamma^{\alpha}_{\alpha\lambda} - \Gamma^{\lambda}_{\mu\alpha}\Gamma^{\alpha}_{\mu\lambda}\right] =$$

$$= g^{\theta\theta}\Big[\overbrace{\partial_\theta\Gamma^{\theta}_{\theta\theta} + \partial_\phi\Gamma^{\phi}_{\theta\theta}} - \partial_\theta\Gamma^{\theta}_{\theta\theta} - \partial_\theta\Gamma^{\phi}_{\phi\theta} +$$
$$+ \cancel{\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\theta\theta}} + \cancel{\Gamma^{\phi}_{\theta\theta}}\Gamma^{\phi}_{\phi\theta} + \Gamma^{\phi}_{\theta\theta}\Gamma^{\theta}_{\theta\phi} + \Gamma^{\phi}_{\theta\theta}\Gamma^{\phi}_{\phi\phi} -$$
$$- \cancel{\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\theta\theta}} - \Gamma^{\theta}_{\theta\phi}\Gamma^{\phi}_{\theta\theta} - \Gamma^{\phi}_{\theta\theta}\Gamma^{\theta}_{\theta\phi} - \Gamma^{\phi}_{\theta\phi}\Gamma^{\phi}_{\theta\phi}\Big] +$$

$$+ g^{\phi\phi}\Big[\partial_\theta\Gamma^{\theta}_{\phi\phi} - \partial_\theta\Gamma^{\phi}_{\phi\phi} - \partial_\phi\Gamma^{\theta}_{\theta\phi} - \partial_\phi\Gamma^{\phi}_{\phi\phi} +$$
$$+ \Gamma^{\theta}_{\phi\phi}\Gamma^{\theta}_{\theta\theta} + \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta} + \Gamma^{\phi}_{\phi\phi}\Gamma^{\theta}_{\theta\phi} + \cancel{\Gamma^{\phi}_{\phi\phi}\Gamma^{\phi}_{\phi\phi}} -$$
$$- \Gamma^{\theta}_{\phi\theta}\Gamma^{\theta}_{\phi\theta} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta} - \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi} - \cancel{\Gamma^{\phi}_{\phi\phi}\Gamma^{\phi}_{\phi\phi}}\Big]$$

$$R = \overbrace{g^{\theta\theta}}^{1/a^2}\left[-\partial_\theta\left(\frac{\cos\theta}{\sin\theta}\right) - \cot an^2\theta\right] +$$
$$+ \frac{1}{a^2\sin^2\theta}\left[-\partial_\theta(\sin\theta\cos\theta) - (-\sin\theta\cos\theta)\frac{\cos\theta}{\sin\theta}\right] = +\frac{1}{a^2} + \frac{1}{a^2} = \frac{2}{a^2}$$

$\Downarrow$

$$\boxed{\text{Ricci Scalar} \equiv \frac{2}{a^2}}$$