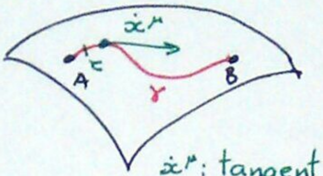
THE GEODESIC EQUATION

Variational principle

In order to measure distances on the surface:



$\dot{x}^\mu$: tangent vector

Curve γ of length $\mathcal{L}$

τ: affin parameter of the curve γ.

Coord. system $\{x^\mu\}$, $x^\mu(\tau)$

$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \quad ; \quad g_{\mu\nu} = g_{\mu\nu}(x)$

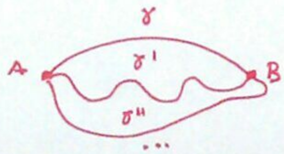
The total length from A to B:

$$\mathcal{L} = \int \sqrt{(dl)^2} \quad [1]$$

Since we're measuring distances: $(dl)^2 = g_{\mu\nu}\, dx^\mu dx^\nu$ [2]

In addition: $dx^\mu = \frac{dx^\mu}{d\tau} d\tau \equiv \dot{x}^\mu d\tau$ [3]

There are infinite paths to go from A to B



however we're looking for the shortest path, distance $\mathcal{L}$, which satisfy $\delta\mathcal{L} = 0$ [4]

[2] $\rightarrow$ [1]: $\mathcal{L} = \int \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu} \underset{[3]}{\overset{\uparrow}{=}} \int \sqrt{g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu}\, d\tau$

[4]: $\delta\mathcal{L} = \int \delta\left(\sqrt{g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu}\right) d\tau = 0$

$$\delta\left(\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\right) = \frac{1}{2}\frac{1}{\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\,\delta\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)$$

$$\delta\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right) = \left(\delta g_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu + \underbrace{g_{\mu\nu}\left(\delta\dot{x}^\mu\right)\dot{x}^\nu + g_{\mu\nu}\dot{x}^\mu\left(\delta\dot{x}^\nu\right)}_{=2g_{\mu\nu}(\delta\dot{x}^\mu)\dot{x}^\nu}$$

(because $g_{\mu\nu} = g_{\nu\mu}$)

Bear in mind: $\delta\dot{x}^\mu = \frac{d}{d\tau}\delta x^\mu$

$\delta g_{\mu\nu} = (\partial_\sigma g_{\mu\nu})\,\delta x^\sigma$

Therefore,

$$0 = \delta \mathcal{L} = \frac{1}{2} \int \frac{1}{\sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \left[(\partial_\sigma g_{\mu\nu}) \delta x^\sigma \dot{x}^\mu \dot{x}^\nu + 2 \underbrace{g_{\mu\nu} \left(\frac{d}{d\tau} \delta x^\mu \right) \dot{x}^\nu}_{} \right] d\tau$$

We integrate by parts:

$$\frac{d}{d\tau} \left[g_{\mu\nu} (\delta x^\mu) \dot{x}^\nu \right] = \frac{d}{d\tau} (g_{\mu\nu} \dot{x}^\nu) \delta x^\mu + \underbrace{g_{\mu\nu} \left(\frac{d}{d\tau} \delta x^\mu \right) \dot{x}^\nu}_{}$$

Then,

$$0 = \delta \mathcal{L} = \frac{1}{2} \int \frac{1}{\sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \left[(\partial_\sigma g_{\mu\nu}) \delta x^\sigma \dot{x}^\mu \dot{x}^\nu + 2 \left(- \frac{d}{d\tau} (g_{\mu\nu} \dot{x}^\nu) \delta x^\mu \right) d\tau + \right.$$

$$+ \frac{1}{2} \int \frac{1}{\sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \left[2 \frac{d}{d\tau} \left(g_{\mu\nu} (\delta x^\mu) \dot{x}^\nu \right) \right] d\tau$$

If we think of a particle moving along any curve from A to B (regardless it's free falling or not), we can find a frame (INERTIAL REFERENCE FRAME) such that $\dot{x}^\mu = (1, \underline{0})$. Hence, $\dot{x}_\mu \dot{x}^\mu = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 1$.

That yields:

$$0 = \delta \mathcal{L} = \frac{1}{2} \int \left[(\partial_\sigma g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu \delta x^\sigma - 2 \frac{d}{d\tau} (g_{\mu\nu} \dot{x}^\nu) \delta x^\mu \right] d\tau +$$

dumbed index $\mu \leftrightarrow \sigma$

$$+ \underbrace{\frac{1}{2} \int 2 \frac{d}{d\tau} \left[g_{\mu\nu} \dot{x}^\nu \delta x^\mu \right] d\tau}_{\int \text{TOTAL DERIVATIVE} = 0} \quad \left(\text{Since } \delta x^\mu \big|_{A \text{ or } B} = 0 \right)$$

Thus,

$$0 = \delta \mathcal{L} = \frac{1}{2} \int \left[(\partial_\sigma g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu - 2 \frac{d}{d\tau} (g_{\sigma\nu} \dot{x}^\nu) \right] \delta x^\sigma d\tau$$

$$\underset{\text{implies}}{\Longrightarrow} \quad (\partial_\sigma g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu - 2 \frac{d}{d\tau} (g_{\sigma\nu} \dot{x}^\nu) = 0$$

Rearranging some terms:

$$\nu \leftrightarrow \alpha$$

$$(\partial_\sigma g_{\mu\nu})\,\dot{x}^\mu \dot{x}^\nu - 2\underbrace{\left(\frac{d}{d\tau} g_{\sigma\nu}\right)}_{\partial_\mu g_{\sigma\nu}\dot{x}^\mu}\dot{x}^\nu - 2 g_{\sigma\nu}\,\ddot{x}^\nu = 0$$

$$2 g_{\sigma\alpha}\,\ddot{x}^\alpha + \Big(\underbrace{2\,\partial_\mu g_{\sigma\nu}}_{\partial_\mu g_{\sigma\nu} + \partial_\mu g_{\nu\sigma}} - \partial_\sigma g_{\mu\nu}\Big)\dot{x}^\mu \dot{x}^\nu = 0$$

(because $g_{\sigma\nu} = g_{\nu\sigma}$)

Multiplying by $\frac{1}{2} g^{\sigma\alpha}$ $\quad (g^{-1}_{\mu\nu} = g^{\mu\nu})$

$$\ddot{x}^\alpha + \frac{1}{2} g^{\sigma\alpha}\left(\partial_\mu g_{\sigma\nu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}\right)\dot{x}^\mu \dot{x}^\nu = 0$$

The Levi-Civita connection:

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma}\left[\partial\, g_{\sigma\beta} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}\right]$$

Then,

$$\boxed{\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu = 0}$$

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