QFT OVERVIEW

— kinetic field theory —

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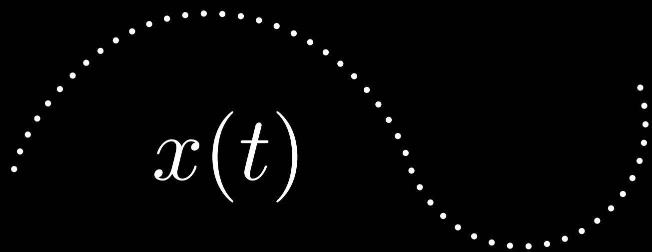
CONTENTS

1. QFT BASICS

TRAJECTORIES

- Classic theory: particles with deterministic trajectories
- Quantum theory: fields and their evolution in time



$$x(t)$$

$$x_a \rightarrow x_b$$

$$\varphi(x, t)$$

$$\varphi_a \rightarrow \varphi_b$$

E.O.M and evaluate at *t*

Transition probability

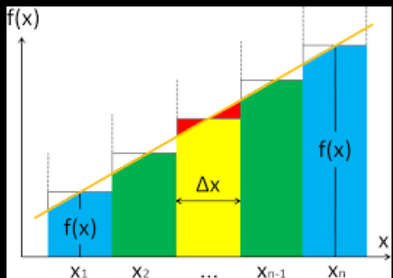
PATH INTEGRALS

- Ordinary integrals

Function $f(x)$

Domain: $x \in [a, b]$

$$\int_a^b \mathrm{d}x \, f(x)$$



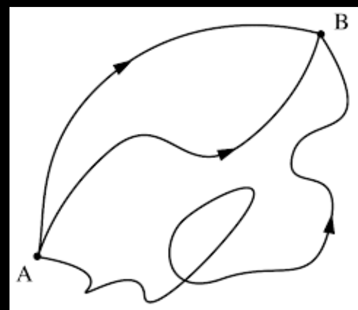
Area under the curve

- Path integrals

Functional $F[\varphi(x)]$

Domain: subspace of $\varphi(x)$

$$\int \underbrace{\mathrm{D}\varphi}_{measure} \; F[\varphi(x)]$$



Action Principle

FUNCTIONAL DERIVATIVES

- Ordinary derivative

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}$$

change w.r.t. *x*

- Functional derivative

$$\frac{\delta F[\varphi(x)]}{\delta\varphi}$$

change w.r.t. the field.

Note: $$\frac{\mathrm{d}x^i}{\mathrm{d}x^j} = \delta^i_j$$

Note: $$\frac{\delta\varphi(x)}{\delta\varphi(x')} = \delta_D(x - x')$$

Properties: linear, product rule, chain rule.

THE GENERATING FUNCTIONAL

- The GF is the generator of all correlation functions

$$Z[\mathbf{J}] = \int \mathrm{d}\Gamma \mathrm{e}^{i \int_0^\infty \mathrm{d}t' \langle \mathbf{J}(t'), \mathbf{x}(t') \rangle}$$

with ***J*** the source, ***x*** set of N particle trajectories, $\langle \cdot, \cdot \rangle$ the scalar product, $\mathrm{d}\Gamma = \mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p}P(\mathbf{q},\mathbf{p})$ the measure, and *P(**q**,**p**)* the initial distribution.

- *Partition function* in Statistical Mechanics: $Z = \sum_i \mathrm{e}^{-\beta E_i}$

CORRELATION FUNCTIONS

- 2PF is the amplitude for a particle to propagate from *x* to *y*:

$$\langle \varphi(x)\varphi(y)\rangle := \langle 0|T\varphi(x)\varphi(y)|0\rangle = \frac{1}{Z_0}\left(\frac{\delta}{i\delta\mathbf{J}(x)}\right)\left(\frac{\delta}{i\delta\mathbf{J}(y)}\right)Z[\mathbf{J}]\Big|_{\mathbf{J}=0}$$

with *T* the time-ordering operator, $|0\rangle$ the vacuum state and Z_0 is a normalisation factor.

- *Statistical mechanics:* statistical correlation between random variables.

DENSITY CORRELATORS

- Number density (N-particle contribution) $$\rho(\mathbf{q},t) = \sum_{i=1}^{N} \rho_i(\mathbf{q},t) = \sum_{i=1}^{N} \delta_D(\mathbf{q} - \mathbf{q}_i(t))$$

- One-particle contribution in Fourier space $$\rho_i(k_1, t_1) = \mathrm{e}^{-i\mathbf{k}_1 \cdot \mathbf{q}_i(t)}$$

- In QFT we work with operators $$\hat{\rho}_i(k_1, t_1) = \mathrm{e}^{-i\mathbf{k}_1^\top \cdot \frac{\delta}{i\delta \mathbf{J}_{q_i}(k_1, t_1)}}$$

- Spatial derivatives generate translations $$\hat{\rho}_i(k_1, t_1) Z[\mathbf{J}] = Z[\mathbf{J} + \mathbf{L}_i(k_1, t_1)]$$

- The density power spectrum or 2-point density correlator

$$G_{\rho\rho}(k_1, k_2, t_1, t_2) := \sum_{i,j=1}^{N} \hat{\rho}_i(k_1, t_1) \hat{\rho}_j(k_2, t_2) Z[\mathbf{J}] \Big|_{\mathbf{J}=0} = \sum_{i,j=1}^{N} Z[\mathbf{L}]$$

FORCE TERM

- Taking the GF at ***L*** and integrating over time

$$Z[\mathbf{L}] = \int \mathrm{d}\Gamma \mathrm{e}^{i\langle \mathbf{L}_q, \mathbf{q}\rangle + i\langle \mathbf{L}_p, \mathbf{p}\rangle - \bar{F}(t)}$$

  with shift tensors

$$\mathbf{L}_q := -\underline{\mathrm{k}}_1 \otimes (\underline{\mathrm{e}}_1 - \underline{\mathrm{e}}_2), \qquad \mathbf{L}_p(t) := g_{qp}(t,0)\mathbf{L}_q$$

  with $\otimes$ tensor product, $\{\underline{\mathrm{e}}_i\}$ basis and $g_{qp}(t,0)$ propagator.

- Force term or time-integrated interaction term

$$\bar{F}(t) = i \int_0^t \mathrm{d}t' \langle \mathbf{L}_p(t'), \nabla \mathbf{V}(t')\rangle$$

- By setting the force term to zero, free GF $Z_0[\mathbf{L}]$

2.PARTICLE TRAJECTORIES

Lagrangian

- The action and the Lagrangian for classical point particles in an expanding universe — q co-moving coordinate —

$$S = \int \mathrm{d}t \mathcal{L}(q, \dot{q}, t) \qquad \mathcal{L}(q, \dot{q}, t) = \frac{m}{2} a^2 \dot{q}^2 - m\phi$$

- Dimensionless time — $\tau = 0$ CMB release —

$$t \longrightarrow \tau = D_+ - D_+(0)\,, \quad \frac{\mathrm{d}}{\mathrm{d}t} = H D_+ f \frac{\mathrm{d}}{\mathrm{d}\tau} \quad \text{with} \quad f := \frac{\mathrm{d}\ln D_+}{\mathrm{d}\ln a} \quad H = \dot{a}/a$$

- Action unchanged, therefore

$$\boxed{\mathcal{L}(q, \dot{q}, \tau) = \frac{g(\tau)}{2} \dot{q}^2 - \nu(q, \tau)}$$

with $g(\tau) := a^2 D_+ f H / H_0$ and $\nu(q, \tau) := a^2 \phi / H_0^2 g(\tau)$

Hamiltonian equations

- Canonically conjugate momentum $$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = g(\tau)\dot{q}$$
- Hamiltonian operator — *g ≈ mass* — $$\mathcal{H} = p \cdot \dot{q} - \mathcal{L} = \frac{p^2}{2g(\tau)} + \nu(q,\tau)$$
- Hamiltonian EOM *expanding space-time*

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}\,, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

compact notation $$\underline{\mathrm{x}} = (q,p) \longrightarrow \partial_t \underline{\mathrm{x}} = \mathcal{J}\partial_x \mathcal{H} = \mathcal{K}\underline{\mathrm{x}}$$

Symplectic matrix $\mathcal{J} = \begin{pmatrix} 0 & \mathbb{1}_3 \\ -\mathbb{1}_3 & 0 \end{pmatrix}$ and $\mathcal{K} = \begin{pmatrix} 0 & \frac{1}{g(\tau)}\mathbb{1}_3 \\ 0 & 0 \end{pmatrix}$ Force matrix

ν = 0

Green's functions— inertial trajectories

- Free & homogeneous solution

$$[\partial_t - \mathcal{K}]\,\underline{\mathrm{x}} = 0$$

$$\underline{\mathrm{x}}_0 = G_R(\tau, 0)\underline{\mathrm{x}}^{(0)}$$

with *retarded* Green's function

$$G_R(\tau, \tau') = \begin{pmatrix} \mathbb{1}_3 & g_{qp}(\tau, \tau')\mathbb{1}_3 \\ 0 & \mathbb{1}_3 \end{pmatrix}$$

$$g_{qp}(\tau, 0) := \int_0^\tau d\tau' \frac{\mathrm{d}\tau'}{g(\tau)}$$

- Free co-moving coord.

$$\boxed{q_0 = q^{(0)} + g_{qp}(\tau, 0)p^{(0)}}$$

Deviates from the interacting trajectories

Green's functions— effective force

- Zel'dovich approximation $\boxed{q(\tau) = q^{(0)} + \tau p^{(0)}}$
- General approach
  - ⋆ Rewrite the Hamiltonian $\mathcal{H} = \frac{p^2}{2} + h(\tau)\frac{p^2}{2} + \nu(q,\tau)$ $h(\tau) = 1/g(\tau) - 1$
  - ⋆ Inhomogeneous EOM *"static universe"*

$$\boxed{[\partial_t - \mathcal{K}_s]\,\underline{\mathrm{x}} = y(\tau)} \qquad \mathcal{K}_s = \begin{pmatrix} 0 & \mathbb{1}_3 \\ 0 & 0 \end{pmatrix} \qquad y(\tau) = \begin{pmatrix} h(\tau)p \\ -\nabla_q \nu \end{pmatrix}$$

  - ⋆ Interacting & inhomogeneous solution

$$\underline{\mathrm{x}}(\tau) = \bar{G}_R(\tau,0)\underline{\mathrm{x}}^{(0)} + \int_0^\tau \mathrm{d}\tau' \bar{G}_R(\tau',0)y(\tau') \qquad \boxed{q(\tau) = q^{(0)} + \tau p^{(0)} + \int_0^\tau \mathrm{d}\tau' g_{qp}(\tau,\tau')\underline{\mathrm{f}}(\tau')}$$

  - ⋆ Green's function

$$\bar{G}_R(\tau,\tau') = \begin{pmatrix} \mathbb{1}_3 & g_{qp}(\tau,\tau')\mathbb{1}_3 \\ 0 & \mathbb{1}_3 \end{pmatrix} \qquad g_{qp}(\tau,\tau') = \tau - \tau'$$

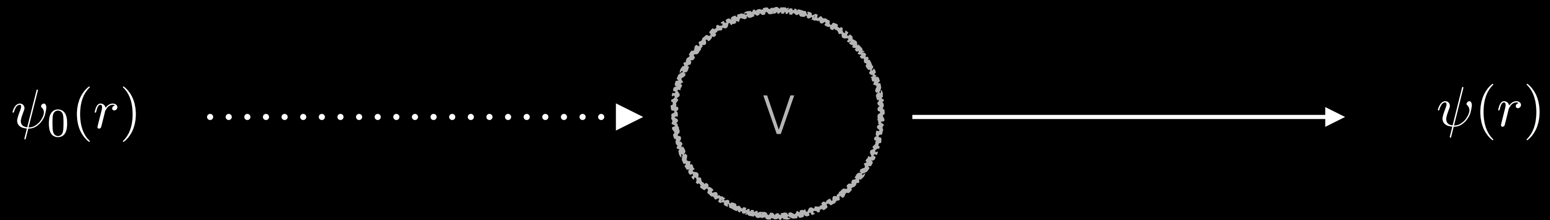
  - ⋆ Effective force $\boxed{\underline{\mathrm{f}}(\tau') := \dot{h}(\tau)p - \nabla_q \nu / g(\tau)}$

3. THE POWER SPECTRUM

KFT Born's approximation

Parenthesis — what is the Born approximation?

Scattering amplitudes



$$\psi(\underline{r}) = \underbrace{e^{i\underline{k}\cdot\underline{r}}}_{incident} + \int d^3\underline{r}' G(\underline{r}-\underline{r}')V(\underline{r}')\,\underbrace{\psi(\underline{r}')}$$

Born approximation *(weak potential)*

$$\psi^{(1)}(\underline{r}) = e^{i\underline{k}\cdot\underline{r}} + \int d^3\underline{r}' G(\underline{r}-\underline{r}')V(\underline{r}')\,\underbrace{e^{i\underline{k}\cdot\underline{r}'}}$$

Interaction potential

- Recall the force term.
- Interaction between two arbitrary particles

$$\nabla_{q_1} V(q_2,t) = i \int \frac{\mathrm{d}^3k}{(2\pi)^3} \underline{\mathrm{k}} \tilde{\nu}(k,t) \mathrm{e}^{i\underline{\mathrm{k}}\cdot[q_1(t)-q_2(t)]} \qquad \tilde{\nu}(k,t) = A(t)/k^2$$

- Replace ***q*** by the **inertial trajectories** and average over ensemble

$$\langle \bar{F}(t) \rangle = \int_0^t \mathrm{d}t' \langle F(t,t') \rangle \qquad \boxed{F(t,t') = 2g_{qp}(t,t')A(t') \left[1 + k_1^3 \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\underline{\mathrm{k}}_1 \cdot \underline{\mathrm{k}}}{k^2} \bar{P}_{\delta\delta}(k,t')\right]}$$

- Damped evolved density power spectrum

$$\boxed{\bar{P}_{\delta\delta}(k,t') := \mathrm{e}^{Q_D(t',k)} g_{qp}^2(t',0) P_{\delta\delta}(k,t')}$$

- Diffusion term

$$Q_D(t',k) := \frac{\sigma_1^2}{3} \Sigma_{i\neq j} L_{p_i} \cdot L_{p_j}$$

with velocity fluctuation amplitude

$$\sigma_1^2 = \int \frac{\mathrm{d}^3k}{(2\pi)^3} k^2 P_{\psi\psi}(k)$$

$$p = \nabla\psi$$

Non-linear power spectrum

- Factorise the GF *Z[L]* …
- The density power spectrum

$$G_{\rho\rho}(k,t) = \mathrm{e}^{Q_D(k,t)-\langle \bar{F}(t)\rangle} \left[(2\pi)^3\delta_D(k) + g_{qp}^2(t,0)P_{\delta\delta}(k,t)\right]$$

- The closed analytic expression for the evolved **non-linear density-fluctuation power spectrum** including particle interactions

$$\bar{P}(k,t) = \mathrm{e}^{Q_D(k,t)-\langle \bar{F}(t)\rangle} \int \mathrm{d}^3q \left[\mathrm{e}^{-g_{qp}^2(t,0)a_{||}(q)k^2} - 1\right] \mathrm{e}^{i\underline{k}\cdot\underline{q}}$$

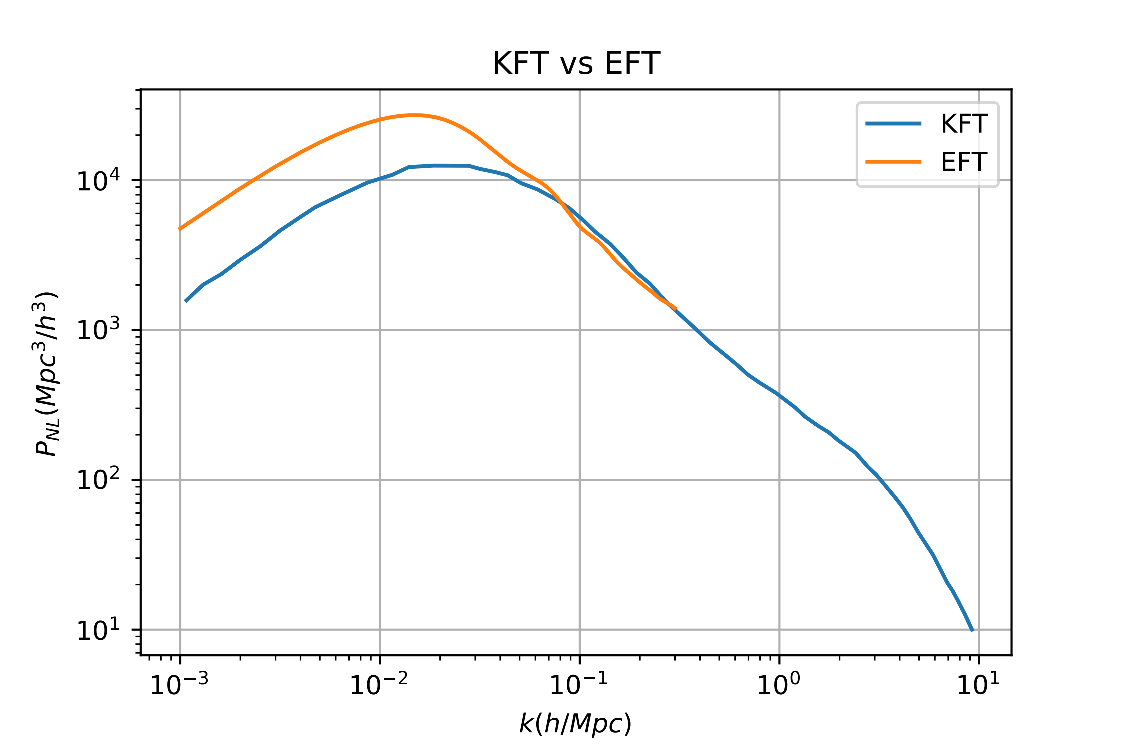
with the projector along trajectories $a_{||} = \mu^2\xi''_{\psi\psi}(q) + (1-\mu^2)\xi'_{\psi\psi}(q)/|q|^2$ and μ = l.o.s

4.CONCLUSIONS

Based on M. Bartelmann et al. work.

KFT	SPT or EFT
Structure formation from small to large scales	From large to small
The Born ansatz gives a simple closed expression for the non-linear power spectrum	EFT encodes all non-linear physics in a set of free-fitting parameters, counter-terms
Phase-space trajectories cannot cross due to the symplectic structure of the Hamilton equations	Shell-crossing occur since SPT is based on either Boltzmann eq. or hydrodynamic eq. — assuming uniquely value velocity fields
No functional determinants singularities — it is unity in phase-space	LPT present functional determinants developing singularities in convergent flows
Full hierarchy of initial momentum correlations — avoids taking moments over momentum space and can track the formation of vorticity	Deciding at which order the series expansion should be truncated is not evident
Non-linear predictions $k \approx 1\text{-}10h/Mpc$	Non-linear predictions $k \approx 0.1\text{-}0.75h/Mpc$

KFT VS EFT



Bartelmann *et al.* 1710.07522

de la Bella *et al.* 1704.05309

Perturbative ansatz	Born ansatz
Results P1h — inner structure	Results mix P1h&P2h
Convergence issues	None
NL=-ve & +ve mode transport from large to small scales	Unobserved
Unclear how terms leading the NL contributions in the perturbative ansatz could be directly compared to those in the Born ansatz	