$f(R)$ Theory of Gravity

Metric formalism

Lucía Fonseca de la Bella

supervised by

C. Marinoni (CPT)

Aix-Marseille Université, AMU

Thursday, 28^{th} November 2013

Outline

Table of Contents

Historical motivation

1. General Relativity Theory (GR) has some theoretical limitations.
2. The Einstein-Hilbert action, S_{EH}, is NOT renormalizable $\Rightarrow$ NOT conventionally quantized.
3. The aim is to modify NOT the structure of GR BUT the field equations of the theory.

Higher order gravity theories $\Rightarrow$ modification of S_{EH} with higher order curvature invariants.

1. Relevant only in very strong gravity regimes such as Planck scale (early universe, inflation, black holes...)
2. Not expected that corrections could affect at low energies (late universe, dark energy).

Table of Contents

Action and f(R) theories

$$S_{EH} = \frac{1}{2\kappa}\int \sqrt{-g} R d^4x \rightarrow S = \frac{1}{2\kappa}\int \sqrt{-g} f(R) d^4x$$

where $\kappa = \frac{8\pi G}{c^4}$, R is the Ricci scalar and g is the determinant of the metric. $f(R)$ theory can avoid the fatal Ostrogadski instability.

The expression for the function $f(R)$ is unknown,

$$f(R) = ... + \frac{\alpha_2}{R^2} + \frac{\alpha_1}{R} - 2\Lambda + R + \frac{R^2}{\beta_2} + \frac{R^3}{\beta_3} + ...$$

Starobinski's inflation

$$f(R) = R - R^2$$

Early universe : $R \rightarrow \infty$ and $f(R) \approx -R^2$.
Late universe : $R \rightarrow 0$ and $f(R) \approx R$.

GENERAL RELATIVITY

$$S_{EH} = \frac{1}{2\kappa} \int \sqrt{-g} R d^4x$$

General Relativity Theory with ordinary matter and radiation is also a $f(R)$ theory

$$f(R) = R - 2\Lambda$$

nevertheless, it is not able to explain the mechanism of the accelerated expansion of the universe.

1. Fine tuning problem Why is the value of the cosmological constant so small ?
2. Coincidence problem. Why "today" is so special, i.e. $\rho_{\Lambda,o} \equiv \rho_{matter,o}$?

Field equation

Varying the action $S = S_{grav} + S_{Matter}$ with respect to the metric $g_{\mu\nu}$ yields the field equations

$$G_{\mu\nu} - \frac{1}{2}\frac{f(R)-f'(R)R}{f'(R)}g_{\mu\nu} - \frac{1}{f'(R)}[\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]f'(R) = \frac{\kappa}{f'(R)}T_{\mu\nu}$$

being $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, $\Box = \nabla_\mu\nabla^\mu$ D'Alembert operator and $f'(R) = \frac{df(R)}{dR}$. Standard Einstein's equations are recovered when $f(R) = R$.

Matter conservation is satisfied : $\boxed{T^{\mu\nu}{}_{;\nu} = 0}$

Interpretation : $\boxed{G_{\mu\nu} = \frac{\kappa}{f'(R)}(T_{\mu\nu} + T^{eff}_{\mu\nu})}$

New contribution to the stress-energy tensor, $T^{eff}_{\mu\nu}$, and the coupling constant is modified by $1/f'(R)$.

f(R) gravity & scalar-tensor theory

Let $V(\phi)$ be the Legendre transform of f(R), i.e.

$\boxed{f(R) = \phi R - V(\phi)}$ where $\phi = f'(R)$.

The scalar-tensor action takes the following form :

$$\boxed{S = \frac{1}{2\kappa}\int \sqrt{-g}[\phi R - V(\phi)]d^4x + S_{matter}}$$

Euler-Lagrange equations read now as

$$V'(\phi) = R$$

$$\phi G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}V(\phi) - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]\phi = \kappa T_{\mu\nu}$$

Computing ϕ and $V(\phi)$, the f(R) field equations are recovered. Therefore, both theories are equivalents.

Field equation

The trace of the field equations is given by

$$f'(R)R - 2f(R) + 3\Box f'(R) = \kappa T$$

which relates R to T differentially and not algebraically like GR does.

Propagating scalar field : $\phi \equiv f'(R)$ (scalar degree of freedom).

De Sitter space is a maximally symmetric empty space, $T_{\mu\nu} = 0$ and $R = const$. The trace equation

$$Rf'(R) - 2f(R) = 0$$

leads to $R = \frac{2f(R)}{f'(R)}$ and $f(R) \sim R^2$. Moreover, the scalar field is given by $\phi = \frac{2V(\phi)}{V'(\phi)}$.

Table of Contents

Modified Friedmann equations

Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2(\theta)d\phi)^2]$$

where (t, r, θ, ϕ) are comoving coordinates, $k = -1, 0, 1$ and $a(t)$ is the scale factor.
Perfect fluid description for matter

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}$$

being u_μ the 4-velocity of a comoving observer, ρ the energy density and P the pressure of the fluid.

Modified Friedmann equations

From observational data, it is possible to make the choice $\boxed{k=0}$.

Friedmann equations (FE) read as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3f'}\left[\rho - \frac{f-f'R}{2} - 3H\dot{R}f''\right]$$
$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6f'}\left[\rho + 3P + f - f'R + 3\dot{R}^2 f''' + 3H\dot{R}f'' + 3\ddot{R}f''\right]$$

where $\dot{} = \frac{d}{dt}$.

Ricci scalar (flat space) read as $\boxed{R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)}$, hence these equations are not easy to integrate.

Assumptions : $f'(R) > 0$ in order to have a positive gravitational coupling and $f''(R) > 0$ to avoid Dolgov-Kawasaki instability.

Table of Contents

Starobinski's model

Further conditions to be satisfied by $f(R)$ models :

1. Not to destroy previous successes of present and early universe cosmology.
2. Explain late universe evolution (Λ, dark energy).
3. Smooth transition between cosmological eras.
4. Correct weak field limit, i.e. the model must explain large scales but also Solar system scales.

Starobinski's model

The f(R) function of the Starobinski's model is given by

$$\boxed{f(R) = R + \mu R_o[(1 + \frac{R^2}{R_o^2})^{-n} - 1]}$$

where the free parameters μ, $n > 0$ and R_o is of the order of the currently observed Λ.

Early universe $R \to \infty$ and $f(R) \approx R - \mu R_o + \mu \frac{R_o^{2n+1}}{R^{2n}}$. The region of validity is $R_o < R < \infty$, singularity at $R = \infty$.
Late universe $R \to 0$ and $f(R) \approx R$.

This model gives rise to viable cosmology and satisfies the Solar system tests. But there are some divergencies in the study of the cosmological perturbations.


THANK YOU!
MERCI! ¡GRACIAS! GRAZIE!
CẢM ƠN! شكرا 谢谢