

THE POINCARÉ CONJECTURE

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THE POINCARÉ CONJECTURE

○ INTRODUCTION

- Motivation of the Poincare conjecture
- Steps to prove the conjecture.

○ THE POINCARÉ CONJECTURE TURNS INTO THEOREM.

- The definition of the Ricci flow.
- The Ricci flow with surgery.

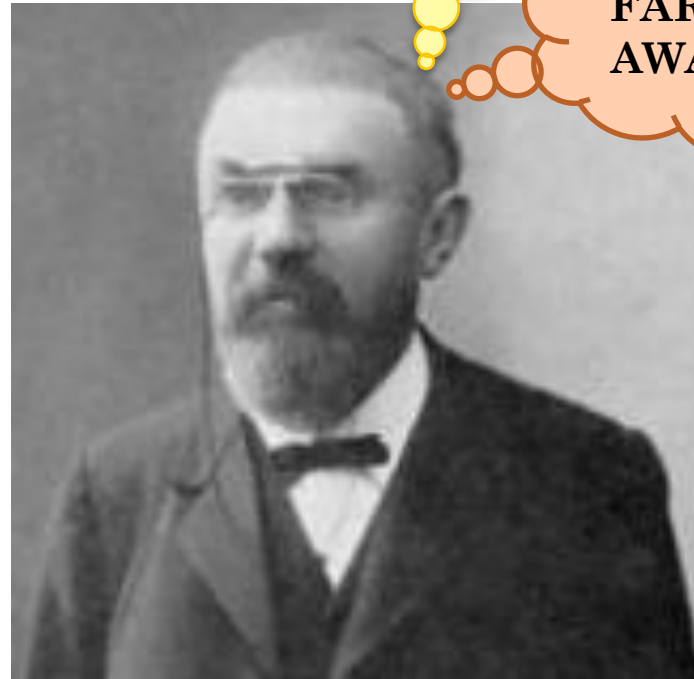
○ IMPACT OF THE PROOF OF THE CONJECTURE.

- References



INTRODUCTION- MOTIVATION OF THE POINCARÉ CONJECTURE

- 1904 (France) Henri Poincaré is studying the stability of the Solar System.
- In this work he was led to the Topology.
- The Poincaré conjecture (P.C.) was the first one in considering the possibility of chaos in a deterministic system, in its work on planetary orbits.



**CRASH
INTO THE
SUN??!!**

**EJECTED
FAR FAR
AWAY??!!**

L. Fonseca de la Bella 2012



INTRODUCTION- MOTIVATION OF THE POINCARÉ CONJECTURE

- Poincaré observed easily that any compact simply connected two-dimensional manifolds are topologically the same as the 2-sphere.*



He conjectured that it is also true in the case of three-dimensional manifolds.



THE POINCARÉ CONJECTURE



INTRODUCTION- STEPS TO PROVE THE CONJECTURE

1904 Poincaré
Conjecture.

many failed attempts become
in very valuable scientific
contributions.

1961 Stephen Smale. Five and
higher-spheres. Field medal.

1970 Michael Freedman . 4-
spheres. Field medal.



INTRODUCTION-

STEPS TO PROVE THE CONJECTURE

Thurston Conjecture. Every compact orientable three-manifold has a canonical decomposition into pieces, which admits a canonical geometric structure from among the 8 maximal simply connected homogeneous Riemannian 3-geometries.

1982 Ricci flow equation. It plays a main role in the demonstration.

2002 Gregory Perelman. Hamilton's disciple. He proved the P.C.
New concept of entropy and the surgery of the Ricci flow.



THE POINCARÉ CONJECTURE TURNS INTO THEOREM.

1. DEFINITION OF THE RICCI FLOW

The Ricci Flow equation:

$$\partial_t g_{ij} = -2Ric_{ij}(g)$$

g is the Riemannian metric tensor and R , the Ricci curvature tensor.

Rewriting the R as function of the Laplacian:

$$\partial_t g = \Delta g$$

(similar to the Fourier equation)

This PDE describe how the curvature disperses in time.*



THE POINCARÉ CONJECTURE TURNS INTO THEOREM.

A Ricci flow: one-parameter family of metrics $g(t)$, parameterized by t in a non-degenerate interval I , on a smooth manifold M .

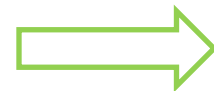
The initial condition: $(M, g(t_0)=g_0)$ (It does not have to exist)

Let g_0 be an Einstein metric:

$$\left. \begin{array}{l} Ric(g_0) = \lambda g_0 \\ g(t) = cg_0 \end{array} \right\} \Rightarrow Ric(g) = \lambda g_0 = \frac{\lambda}{c} g$$

$\lambda, c > 0, const.$

Using that, one can construct the solution



THE POINCARÉ CONJECTURE TURNS INTO THEOREM.

- Let's assume that the *one-parameter family of metrics* $g(t) = u(t) g_0$ is a solution:

$$\left. \begin{aligned} \partial_t g &= u'(t) g_0 \\ \partial_t g &= -2\text{Ric}(u(t)g_0) = -2\text{Ric}(g_0) = -2\lambda g_0 \end{aligned} \right\} u'(t) = -2\lambda \rightarrow u(t) = 1 - 2\lambda t$$

Notice $\rightarrow g(t_0) = u(t_0) g_0 = g_0 \Rightarrow u(t_0) = 1$

- Soliton type solution:** $g(t) = (1 - 2\lambda t)g_0$
It “explodes” for $t = 1/2\lambda$!

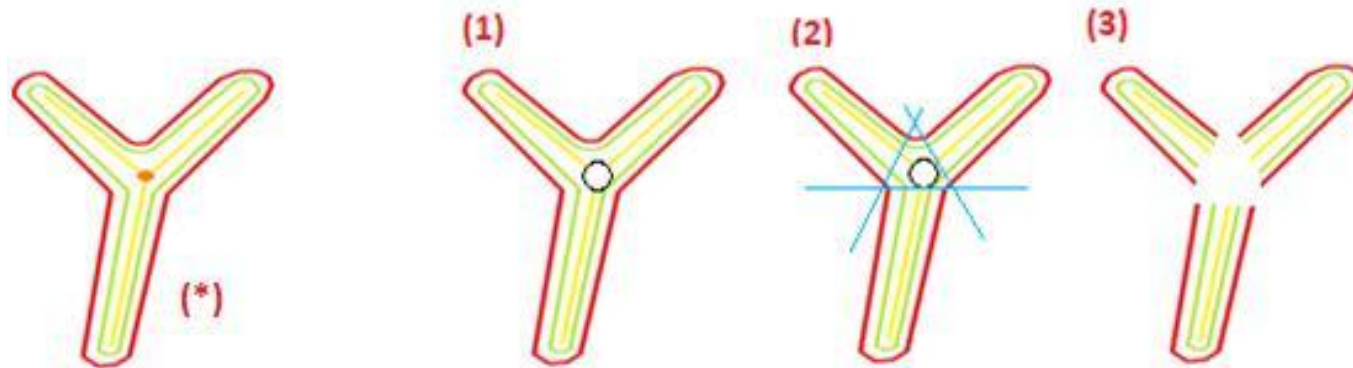
Singularities of the Ricci flow equation.



THE POINCARÉ CONJECTURE TURNS INTO THEOREM.

2. THE RICCI FLOW WITH SURGERY

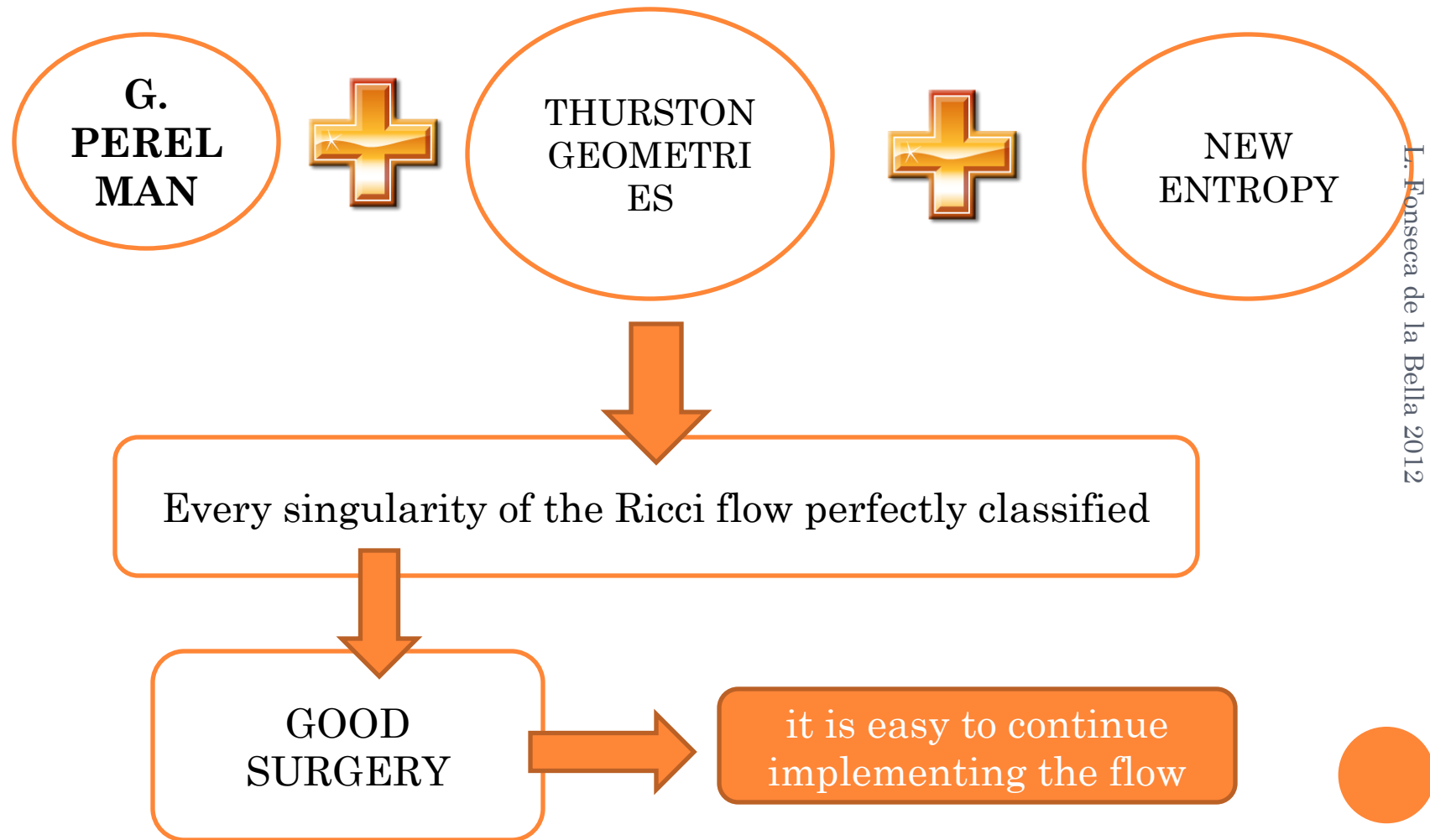
Hamilton introduces the process known as surgery



It allows going on with the implementation of the flow till its extinction in a topological manifold (which satisfies the hypothesis of the P.C) !



THE POINCARÉ CONJECTURE TURNS INTO THEOREM.



THE POINCARÉ CONJECTURE TURNS INTO THEOREM.

- The continued implementation of the flow **before and after each surgery** is carried out by a **scaling operation**.
- One must check in each step that the obtained metric g is a solution of the flow equation.

- **n-th scaling**

Whether higher but finite values of n are taken, one will obtain $g_n = \lambda_n g_0$ “renormalization”



THE POINCARÉ CONJECTURE TURNS INTO THEOREM.

- **Summing up,**
- ❖ $\text{Ric}=\lambda g$
- ❖ Consecutive implementations of the Ricci flow



the final metric corresponds to a 3-sphere which is homeomorphic to the initial manifold.

- ❖ Hence, it fulfills the requisite of being a simply connected 3-manifold.

**G.PERELEMAN PROVED
THEPOINCARÉ CONJECTURE**

IMPACT OF THE PROOF OF THE CONJECTURE.

- The Poincaré Conjecture → one of the Seven Problems of the millennium.
- The Clay Mathematics Institut gives to G. Perelman 1million dollars because of its proof of the P.C.
- Perelman rejected the award and he becomes internationally known.



The Poincaré Conjecture and G.Perelman are famous both in the scientific frame and in the whole world!



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